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A unified recurrence operator method for obtaining normalized explicit wavefunctions for shape-invariant potentials

Chun-Sheng Jia†, Xiao-Guo Wang‡, Xiao-Ke Yao†, Pu-Chun Chen† and Wei Xiao†

† Department of Automation, Southwest Petroleum Institute, Nanchong 637001, People's Republic of China

‡ Department of Bridge and Underground Engineering, Southwest Jiaotong University, Chengdu 610031, People's Republic of China

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Abstract. We construct a unified recurrence operator method for obtaining explicit expressions for the wavefunctions of shape-invariant potentials. It is found that the normalized coefficients for the energy eigenfunctions satisfy a universal recurrence relation. The procedure is illustrated in detail for four potentials. We work out the normalized explicit wavefunctions of Hulthen potential, for which the normalized explicit wavefunctions have not been previously calculated.

1. Introduction

After introducing the concepts of supersymmetry [1] and shape invariance [2] in non-relativistic quantum mechanics, the energy eigenvalues can be worked out algebraically for almost all exact solvable potentials [2–5]. Using operator techniques, Dutt *et al* [6] and Dabrowska *et al* [7] have obtained unnormalized explicit wavefunctions for shape-invariant potentials with a translation of parameters. Recently, Barclay *et al* [4] discovered a large class of new shape-invariant potentials with a scaling ansatz for the change of parameters, and they also obtained unnormalized explicit expressions for the eigenfunctions of these potentials. However, using operator techniques, how to obtain normalized explicit expressions for the eigenfunctions of shape-invariant potentials is still an open question [8]. In view of the fact that shape-invariant potentials have not been exhausted [9], establishing a theory to calculate the normalized explicit wavefunctions for shape-invariant potentials is of considerable interest.

In this paper, we propose a unified recurrence operator method to calculate the normalized explicit wavefunctions for shape-invariant potentials within the framework of supersymmetric quantum mechanics. It has been found that the normalized coefficients for the wavefunctions of shape-invariant potentials satisfy a universal recurrence relation. Using the usual factorization methods, it is very difficult to obtain the normalized explicit eigenfunctions for some shape-invariant potentials (such as the modified Pöschl–Teller potential, the Hulthen potential, etc). However, for these potentials we can also obtain their normalized explicit wavefunctions by using the present recurrence operator method, and not using the special functions. We suggest that the unified recurrence operator method is useful in obtaining the normalized explicit wavefunctions of the shape-invariant potentials with the help of computer software such as *Mathematica*.

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2. The universal recurrence operator method

The one-dimensional stationary Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)\right]\Psi(x)=E\Psi(x) \tag{1}$$

where $\Psi(x)$ is the wavefunction, $V(x)$ the potential and E the energy. The unnormalized ground-state wavefunction $\Psi_0(x)$ is written as

$$\Psi_0(x)=\exp\left(-\frac{\sqrt{2m}}{\hbar}\int W(x)\,\mathrm{d}x\right)=\exp\left(\int Z(x)\,\mathrm{d}x\right) \tag{2}$$

where $W(x)$ is called a superpotential. Substituting equation (2) into (1) gives

$$Z'+Z^2=\upsilon(x)-\varepsilon_0 \tag{3}$$

where $\upsilon(x)=2mV(x)/\hbar^2$, $\varepsilon_0=2mE_0/\hbar^2$ and E_0 is the ground-state energy. Equation (3) is a nonlinear Riccati equation.

In terms of the superpotential $W(x)$, the supersymmetric partner potentials $V_+(x)$ and $V_-(x)$ are given by [3]

$$V_+(x)=W^2(x)+\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W(x)}{\mathrm{d}x} \tag{4}$$

$$V_-(x)=W^2(x)-\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{5}$$

Also, the operators A and A^+ are given by

$$A^+=-\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x}+W(x) \tag{6}$$

$$A=\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x}+W(x). \tag{7}$$

If $V_+(x)$ and $V_-(x)$ have similar shapes, they are said to be shape invariant, and they satisfy the following relation [2]

$$V_+(x,a_0)=V_-(x,a_1)+R(a_1) \tag{8}$$

where a_0 is a set of parameters, a_1 is a function of a_0 ($a_1=f(a_0)$, say) and the remainder $R(a_1)$ is independent of x. The Hamiltonians corresponding to the potentials $V_+(x)$ and $V_-(x)$ are given by

$$H_+(x,a_0)=-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_+(x,a_0) \tag{9}$$

$$H_-(x,a_0)=-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_-(x,a_0). \tag{10}$$

Gendenshtein [2] showed that the energy spectrum of $H_-(x,a_0)$ is given by

$$E_0^{(-)}=0 \qquad E_n^{(-)}=\sum_{k=1}^{n}R(a_k) \qquad a_k=f^k(a_0) \tag{11}$$

while its unnormalized energy eigenfunctions are given by [7]

$$\Psi_{n+1}^{(-)}(x,a_0)=A^+(x,a_0)\Psi_n^{(-)}(x,a_1). \tag{12}$$

The potentials $V(x)$ and $V_-(x, a_0)$ have the following relation [5]:

$$V(x) = V_-(x, a_0) + E_0. \tag{13}$$

Thus, the energy eigenfunctions for the potentials $V(x)$ and $V_-(x, a_0)$ are the same. In all subsequent discussions, we only consider wavefunctions of $V_-(x, a_0)$, so the superscript $(-)$ will be suppressed for simplicity. In order to find recurrence relations for the normalized coefficients, we now carry out the following calculations:

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi_{n+1}^2(x, a_0)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \Psi_{n+1}(x, a_0) A^+(x, a_0)\Psi_n(x, a_1)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi_{n+1}(x, a_0)\left(-\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0)\right)\Psi_n(x, a_1)\,\mathrm{d}x \\
&= -\frac{\hbar}{\sqrt{2m}}\Psi_{n+1}(x, a_0)\Psi_n(x, a_1)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\hbar}{\sqrt{2m}}\Psi_n(x, a_1)\frac{\mathrm{d}\Psi_{n+1}(x, a_0)}{\mathrm{d}x}\,\mathrm{d}x \\
&\quad + \int_{-\infty}^{\infty}\Psi_{n+1}(x, a_0)W(x, a_0)\Psi_n(x, a_1)\,\mathrm{d}x.
\end{aligned}$$

When $x \to \pm\infty$, $\Psi_{n+1}(x, a_0)$ and $\Psi_n(x, a_1)$ are equal to zero, so we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi_{n+1}^2(x, a_0)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \Psi_n(x, a_1)\left[\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0)\right]\Psi_{n+1}(x, a_0)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi_n(x, a_1)\left[\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0)\right]\left[-\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0)\right]\Psi_n(x, a_1)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi_n(x, a_1)\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2(x, a_0) + \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W(x, a_0)}{\mathrm{d}x}\right]\Psi_n(x, a_1)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi_n(x, a_1)H_+(x, a_0)\Psi_n(x, a_1)\,\mathrm{d}x
\end{aligned} \tag{14}$$

where $H_+(x, a_0)$ can be written as

$$\begin{aligned}
H_+(x, a_0) &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x, a_0) = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x, a_1) + R(a_1) \\
&= H_-(x, a_1) + R(a_1).
\end{aligned} \tag{15}$$

Substituting (15) into (14) yields

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi_{n+1}^2(x, a_0)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \Psi_n(x, a_1)[H_-(x, a_1) + R(a_1)]\Psi_n(x, a_1)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi_n(x, a_1)[E_n^{(-)}(a_1) + R(a_1)]\Psi_n(x, a_1)\,\mathrm{d}x \\
&= [E_n^{(-)}(a_1) + R(a_1)]\int_{-\infty}^{\infty}\Psi_n^2(x, a_1)\,\mathrm{d}x.
\end{aligned} \tag{16}$$

Letting $N_{n+1}(a_0)$ and $N_n(a_1)$ be the normalized coefficients for $\Psi_{n+1}(x, a_0)$ and $\Psi_n(x, a_1)$, respectively, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} N_{n+1}^2(a_0)\Psi_{n+1}^2(x, a_0)\,\mathrm{d}x \\
&\qquad = N_{n+1}^2(a_0)[E_n^{(-)}(a_1) + R(a_1)]\frac{1}{N_n^2(a_1)}\int_{-\infty}^{\infty} N_n^2(a_1)\Psi_n^2(x, a_1)\,\mathrm{d}x \\
&\qquad = N_{n+1}^2(a_0)[E_n^{(-)}(a_1) + R(a_1)]\frac{1}{N_n^2(a_1)} = 1.
\end{aligned} \tag{17}$$

This leads to

$$N_{n+1}(a_0) = \frac{N_n(a_1)}{[E_n^{(-)}(a_1) + R(a_1)]^{1/2}}. \tag{18}$$

This is a universal recurrence relation for all the shape-invariant potentials.

With the help of equations (12) and (18), we can work out the normalized explicit wavefunctions for all the known shape-invariant potentials. To clarify the whole procedure, we will explicitly compute the first few normalized energy eigenfunctions for four shape-invariant potentials (Coulomb, Morse oscillator, modified Pöschl–Teller and Hulthen potential (s state)).

3. Applications

3.1. Coulomb potential

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{19}$$

The equivalent potential for the radial motion is given by

$$V_l(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} + \frac{\hbar^2 l(l+1)}{2mr^2}.$$

We obtain the following from [5]

$$E_{n_r}^{(-)}(a_1) + R(a_1) = \frac{me^4}{2\hbar^2}\left[\frac{1}{a_0^2} - \frac{1}{(a_1+n_r)^2}\right] \qquad n_r = 0, 1, 2, \ldots \tag{20}$$

$$A^+(r, a_0) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{a_0}{r} - \frac{\sigma}{2a_0}\right) \tag{21}$$

$$\chi_0(r, a_0) = r^{a_0}\,\mathrm{e}^{(-\sigma/2a_0)r} \tag{22}$$

where $a_0 = l + 1$, $a_1 = a_0 + 1$ and $\sigma = me^2/2\pi\varepsilon_0\hbar^2$. The radial wavefunction $R(r)$ has been written in the form $R(r) = \chi(r)/r$.

From the normalized condition

$$\int_0^\infty N_0^2(a_0)\left(\frac{\chi_0(r, a_0)}{r}\right)^2 r^2\,\mathrm{d}r = 1$$

we obtain the normalized coefficients by using equations (18) and (20):

$$N_0(a_0) = \left[\left(\frac{a_0}{\sigma}\right)^{2a_0+1}\Gamma(2a_0+1)\right]^{-1/2} \tag{23a}$$

$$N_1(a_0) = \left[\left(\frac{a_0+1}{\sigma}\right)^{2a_0+3}\Gamma(2a_0+3)\frac{me^4}{2\hbar^2}\left(\frac{1}{a_0^2} - \frac{1}{(a_0+1)^2}\right)\right]^{-1/2} \tag{23b}$$

$$N_2(a_0) = \frac{2\hbar^2}{me^4}\left[\left(\frac{1}{a_0^2} - \frac{1}{(a_0+2)^2}\right)\left(\frac{1}{(a_0+1)^2} - \frac{1}{(a_0+2)^2}\right)\left(\frac{a_0+2}{\sigma}\right)^{2a_0+5}\Gamma(2a_0+5)\right]^{-1/2}. \tag{23c}$$

The unnormalized wavefunctions are obtained by using equations (12), (21) and (22):

$$\chi_1(r, a_0) = -\frac{\hbar}{\sqrt{2m}}\left[2a_0 + 1 - \frac{\sigma}{2}\left(\frac{1}{a_0+1} + \frac{1}{a_0}\right)r\right]r^{a_0}\,\mathrm{e}^{-(\sigma/2(a_0+1))r} \tag{24a}$$

$$\chi_2(r,a_0)=\frac{\hbar^2}{2m}\left[(2a_0+3)(a_0+1)-(2a_0+3)\frac{\sigma}{2(a_0+2)}r\right.$$
$$-\frac{\sigma}{2}\left(\frac{1}{a_0+2}+\frac{1}{a_0+1}\right)(a_0+2)r+\frac{\sigma}{2}\left(\frac{1}{a_0+2}+\frac{1}{a_0+1}\right)\frac{\sigma}{2(a_0+2)}r^2$$
$$\left.+\left(\frac{a_0}{r}-\frac{\sigma}{2a_0}\right)\left(2a_0+3-\frac{\sigma}{2}\left(\frac{1}{a_0+2}+\frac{1}{a_0+1}\right)r\right)r^{a_0+1}\,\mathrm{e}^{-(\sigma/2(a_0+2))r}\right]. \tag{24b}$$

We put $n=n_r+l+1$. By using equations (23) and (24) and after algebraic simplification, we obtain the normalized wavefunctions

$$R_{1,0}(r)=\frac{1}{a_\mathrm{B}^{3/2}}2\,\mathrm{e}^{-r/a_\mathrm{B}} \tag{25a}$$
$$R_{2,0}(r)=\frac{1}{(2a_\mathrm{B})^{3/2}}\left(2-\frac{r}{a_\mathrm{B}}\right)\mathrm{e}^{-r/2a_\mathrm{B}} \tag{25b}$$
$$R_{2,1}(r)=\frac{1}{2\sqrt{6}a_\mathrm{B}^{3/2}}\frac{r}{a_\mathrm{B}}\mathrm{e}^{-r/2a_\mathrm{B}} \tag{25c}$$
$$R_{3,0}(r)=\frac{1}{(3a_\mathrm{B})^{3/2}}\left[2-\frac{4r}{3a_\mathrm{B}}+\frac{4}{27}\left(\frac{r}{a_\mathrm{B}}\right)^2\right]\mathrm{e}^{-r/3a_\mathrm{B}} \tag{25d}$$

where $a_\mathrm{B}=2/\sigma=4\pi\varepsilon_0\hbar^2/me^2$, which is Bohr radius. These results are the same as those obtained in [10] through the usual factorization method.

3.2. Morse oscillator potential

$$V(x)=U_0[\mathrm{e}^{-2\alpha x}-2\,\mathrm{e}^{-\alpha x}]. \tag{26}$$

Using the factorization method, Nieto and Simmons [11] obtained the exact normalized and closed-form eigenfunctions written in terms of associated Laguerre polynomials

$$\Psi_n(x)=N(n,\lambda)y^{\lambda-(1/2)-n}\mathrm{e}^{-y/2}L_n^{(2\lambda-2n-1)}(y)\qquad 0\leqslant n\leqslant[\lambda-\tfrac{1}{2}] \tag{27}$$

where $\lambda=(2mU_0/\hbar^2a^2)^{1/2}$, $y=2\lambda\,\mathrm{e}^{-\alpha x}$ and

$$N(n,\lambda)=\left[\frac{\alpha(2\lambda-2n-1)\Gamma(n+1)}{\Gamma(2\lambda-n)}\right]^{1/2}. \tag{28}$$

Letting $Z(x)=A\,\mathrm{e}^{-\alpha x}+B$, substituting this into equation (3) yields

$$A=\beta\qquad B=\frac{\alpha}{2}-\beta$$

where $\beta=\sqrt{2mU_0}/\hbar$.

On using equations (4), (5), (8) and (11), we obtain

$$R(a_1)=\frac{\hbar^2}{2m}(a_0^2-a_1^2) \tag{29}$$
$$E_n^{(-)}(a_0)=\frac{\hbar^2}{2m}(a_0^2-a_n^2) \tag{30}$$

where $a_0=(\alpha/2)-\beta$, $a_1=a_0+\alpha$. From equations (29) and (30), we get

$$E_n^{(-)}(a_1)+R(a_1)=\frac{\hbar^2}{2m}[a_0^2-(a_0+n\alpha+\alpha)^2]. \tag{31}$$

The operator $A^+(x, a_0)$ and unnormalized ground-state wavefunction $\Psi_0(x, a_0)$ are given by

$$A^+(x, a_0) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \beta\,\mathrm{e}^{-\alpha x} + a_0\right) \tag{32}$$

$$\Psi_0(x, a_0) = \exp\left(-\frac{\beta}{\alpha}\,\mathrm{e}^{-\alpha x} + a_0 x\right). \tag{33}$$

From

$$\int_{-\infty}^{\infty} N_0^2(a_0)\Psi_0^2(x, a_0)\,\mathrm{d}x = 1$$

we obtain the normalized coefficients for the first three wavefunctions by using equations (18) and (31):

$$N_0(a_0) = \frac{\sqrt{\alpha}}{(2\beta/\alpha)^{a_0/\alpha}[\Gamma(-2a_0/\alpha)]^{1/2}} \tag{34a}$$

$$N_1(a_0) = \frac{1}{(2\beta/\alpha)^{(a_0+\alpha)/\alpha}}\left[\frac{\alpha}{\Gamma(-2(a_0+\alpha)/\alpha)}\right]^{1/2}\left[-\frac{\hbar^2}{2m}(2a_0+\alpha)\alpha\right]^{-1/2} \tag{34b}$$

$$N_2(a_0) = \left(\frac{2\beta}{\alpha}\right)^{-(a_0+2\alpha)/\alpha}\left[-\frac{\hbar^2}{2m}(2a_0+3\alpha)\alpha\right]^{-1/2}\left[-\frac{\hbar^2}{2m}(2a_0+2\alpha)2\alpha\right]^{-1/2} \times\left[\frac{\alpha}{\Gamma(-2(a_0+2\alpha)/\alpha)}\right]^{1/2}. \tag{34c}$$

According to the definition of the gamma function, we have for $N_n(a_0)$

$$a_0 + n\alpha < 0$$

which leads to

$$n < \sqrt{\frac{2mU_0}{\hbar^2\alpha^2}} - \frac{1}{2}.$$

This condition is consistent with the conclusion in the literature [11].

The unnormalized wavefunctions are obtained from equations (12), (32) and (33):

$$\Psi_1(x, a_0) = -\frac{\hbar}{\sqrt{2m}}(2\beta\,\mathrm{e}^{-\alpha x} + 2a_0 + \alpha)\exp\left(-\frac{\beta}{\alpha}\,\mathrm{e}^{-\alpha x} + (a_0+\alpha)x\right) \tag{35a}$$

$$\Psi_2(x, a_0) = \frac{\hbar^2}{2m}[4\beta^2\,\mathrm{e}^{-2\alpha x} + (10\beta\alpha + 8\beta a_0)\,\mathrm{e}^{-\alpha x} + 4a_0^2 + 10a_0\alpha + 6\alpha^2] \times\exp\left(-\frac{\beta}{\alpha}\mathrm{e}^{-\alpha x} + (a_0+2\alpha)x\right). \tag{35b}$$

By using equations (34) and (35) and after algebraic simplification, we obtain the normalized wavefunctions

$$\Psi_0(x) = \left(\frac{\alpha}{\Gamma(2\lambda-1)}\right)^{1/2}(2\lambda)^{(2\lambda-1)/2}\exp\left(-\frac{2\lambda-1}{2}\alpha x - \lambda\,\mathrm{e}^{-\alpha x}\right) \tag{36a}$$

$$\Psi_1(x) = \left(\frac{\alpha}{(2\lambda-2)\Gamma(2\lambda-3)}\right)^{1/2}(2\lambda)^{(2\lambda-3)/2}(2\lambda-2-2\lambda\,\mathrm{e}^{-\alpha x}) \times\exp\left(-\frac{2\lambda-3}{2}\alpha x - \lambda\,\mathrm{e}^{-\alpha x}\right) \tag{36b}$$

$$\Psi_2(x) = \left(\frac{2\alpha}{(2\lambda-3)(2\lambda-4)\Gamma(2\lambda-5)}\right)^{1/2}(2\lambda)^{(2\lambda-5)/2}$$
$$\times[2\lambda^2\,\mathrm{e}^{-2\alpha x} - 2\lambda(2\lambda-3)\,\mathrm{e}^{-\alpha x} + 2\lambda^2 - 7\lambda + 6]$$
$$\times\exp\left(-\frac{2\lambda-5}{2}\alpha x - \lambda\,\mathrm{e}^{-\alpha x}\right) \qquad (36c)$$

where we have used the substitution of $\lambda = \beta/\alpha$. These results are consistent with the results obtained from equation (27).

3.3. Modified Pöschl–Teller potential

$$V(x) = -\frac{\hbar^2}{2m}\lambda(\lambda+1)\mathrm{sech}^2 x. \qquad (37)$$

From [5] we obtain

$$E_n^{(-)}(a_1) + R(a_1) = \frac{2\hbar}{\sqrt{2m}}(n+1)a_1 - \frac{\hbar^2}{2m}(n^2-1) \qquad n = 0, 1, 2, 3, \ldots \qquad (38)$$

$$A^+(x, a_0) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\sqrt{2m}}{\hbar}a_0\tan x\right) \qquad (39)$$

$$\Psi_0(x, a_0) = (\mathrm{sech} x)^{-(\sqrt{2m}/\hbar)a_0} \qquad (40)$$

where $a_0 = \hbar\lambda/\sqrt{2m}$, $a_1 = a_0 - \hbar/\sqrt{2m}$.

From

$$\int_{-\infty}^{\infty} N_0^2(a_0)\Psi_0^2(x, a_0)\mathrm{d}x = N_0^2(a_0)\frac{[2^{(\sqrt{2m}/\hbar)a_0}\Gamma((\sqrt{2m}/\hbar)a_0+1)]^2}{(\sqrt{2m}/\hbar)a_0\Gamma(2(\sqrt{2m}/\hbar)a_0+1)} = 1$$

the normalized coefficients are obtained by using equations (18) and (38)

$$N_0(a_0) = \left[\frac{\sqrt{2m}}{\hbar}a_0\Gamma\left(2\frac{\sqrt{2m}}{\hbar}a_0+1\right)\right]^{1/2}\left[2^{(\sqrt{2m}/\hbar)a_0}\Gamma\left(\frac{\sqrt{2m}}{\hbar}a_0+1\right)\right]^{-1} \qquad (41a)$$

$$N_1(a_0) = \left[\frac{\sqrt{2m}}{\hbar}\left(\frac{\sqrt{2m}}{\hbar}a_0-1\right)\Gamma\left(2\frac{\sqrt{2m}}{\hbar}a_0-1\right)\right]^{1/2}$$
$$\times\left[2^{(\sqrt{2m}/\hbar)a_0-1}\Gamma\left(\frac{\sqrt{2m}}{\hbar}a_0\right)\left(2\frac{\sqrt{2m}}{\hbar}a_0-1\right)^{1/2}\right]^{-1} \qquad (41b)$$

$$N_2(a_0) = (\sqrt{2m}/\hbar)\left[\left(\frac{\sqrt{2m}}{\hbar}a_0-2\right)\Gamma\left(2\frac{\sqrt{2m}}{\hbar}a_0-3\right)\right]^{1/2}$$
$$\times\left[2^{(\sqrt{2m}/\hbar)a_0-2}\left(2\frac{\sqrt{2m}}{\hbar}a_0-3\right)^{1/2}\left[\frac{4\hbar}{\sqrt{2m}}\left(a_0-\frac{\hbar}{\sqrt{2m}}\right)\right]^{1/2}\right.$$
$$\left.\times\Gamma\left(\frac{\sqrt{2m}}{\hbar}a_0-1\right)\right]^{-1} \qquad (41c)$$

where we have used the gamma function, which is defined as $\Gamma(x) = \int_0^\infty t^{x-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$. Using equations (12), (38) and (39), we obtain the unnormalized wavefunctions

$$\Psi_1(x, a_0) = \frac{\hbar}{\sqrt{2m}}\left(2\frac{\sqrt{2m}}{\hbar}a_0-1\right)(\mathrm{sech} x)^{(\sqrt{2m}/\hbar)a_0-1}\tan x \qquad (42a)$$

$$\Psi_2(x,a_0) = \left(2\frac{\sqrt{2m}}{\hbar}a_0 - 3\right)\left[-\frac{\hbar^2}{2m}\left(\frac{\sqrt{2m}}{\hbar}a_0 - 2\right)\operatorname{sech}^{-1}x + \frac{\hbar^2}{2m}\operatorname{sech}x + \frac{\hbar a_0}{\sqrt{2m}}\tan x\right]$$
$$\times(\operatorname{sech}x)^{(\sqrt{2m}/\hbar)a_0-2}\tan x. \quad (42b)$$

By using equations (41) and (42) and after algebraic simplification, we obtain the normalized wavefunctions

$$\Psi_0(x) = \frac{(\lambda\Gamma(2\lambda+1))^{1/2}}{2^\lambda\Gamma(\lambda+1)}\operatorname{sech}^\lambda x \quad (43a)$$

$$\Psi_1(x) = \left(\frac{(\lambda+1)\Gamma(2\lambda)}{2^{\lambda-1}\Gamma(\lambda)}\right)^{1/2}\tan x\operatorname{sech}^{\lambda-1}x \quad (43b)$$

$$\Psi_2(x) = \frac{1}{2^\lambda\Gamma(\lambda)}(2(\lambda-2)(2\lambda-2)(2\lambda-3)\Gamma(2\lambda-3))^{1/2}[(2\lambda-1)\tan^2x - 1]\operatorname{sech}^{\lambda-2}x. \quad (43c)$$

These results are the same as those in [12] expressed in terms of universal associated-Legendre polynomials by using a hyperbolic function transformation.

3.4. *Hulthen potential*

$$V(r) = -\frac{V_0}{\exp(r/a)-1} \qquad V_0, a > 0. \quad (44)$$

The equivalent potential for the radial motion is given by

$$V_l(r) = -\frac{V_0}{\exp(r/a)-1} + \frac{\hbar^2 l(l+1)}{2mr^2}.$$

For the s state ($l = 0$), we obtain the following from [9]

$$E_{n_r}^{(-)}(a_1) + R(a_1) = \frac{\hbar^2}{2m}\left[\left(\frac{a_0^2-\beta}{2a_0}\right)^2 - \left(\frac{(n+1)^2a_1^2-\beta}{2(n+1)a_1}\right)^2\right] \qquad n_r = 0,1,2,\ldots \quad (45)$$

$$A^+(r,a_0) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{a_0}{\mathrm{e}^{\alpha r}-1} + \frac{a_0^2-\beta}{2a_0}\right) \quad (46)$$

$$\chi_0(r,a_0) = (\mathrm{e}^{\alpha r}-1)^{a_0/\alpha}\,\mathrm{e}^{-((a_0^2+\beta)/2a_0)r} \quad (47)$$

where $a_0 = \alpha$, $a_1 = a_0 + \alpha$, $\alpha = 1/a$, and $\beta = 2mV_0/\hbar^2$. $\chi(r)$ is defined as $\chi(r) = rR(r)$.

From the normalized condition

$$\int_0^\infty N_0^2(a_0)\left(\frac{\chi_0(r,a_0)}{r}\right)^2 r^2\,\mathrm{d}r = 1$$

we obtain the normalized coefficients by using equations (18) and (45)

$$N_0(a_0) = \sqrt{\alpha}\left[B\left(\frac{\beta}{2a_0} - \frac{a_0}{\alpha}, \frac{2a_0}{\alpha}+1\right)\right]^{-1} \quad (48a)$$

$$N_1(a_0) = \sqrt{\alpha}\left[\frac{\hbar}{\sqrt{2m}}B\left(\frac{\beta}{\alpha a_0+\alpha^2} - \frac{a_0}{\alpha} - 1, \frac{2a_0}{\alpha}+3\right)\right.$$
$$\left.\times\left[\left(\frac{a_0^2-\beta}{2a_0}\right)^2 - \left(\frac{(a_0+\alpha)^2-\beta}{2(a_0+\alpha)}\right)^2\right]^{1/2}\right]^{-1} \quad (48b)$$

$$N_2(a_0) = \sqrt{\alpha}\left[\frac{\hbar^2}{2m}B\left(\frac{\beta}{\alpha a_0+2\alpha^2}-\frac{a_0}{\alpha}-2,\frac{2a_0}{\alpha}+5\right)\right.$$
$$\times\left[\left(\frac{(a_0+\alpha)^2-\beta}{2a_0+2\alpha}\right)^2-\left(\frac{(a_0+2\alpha)^2-\beta}{2a_0+4\alpha}\right)^2\right]^{1/2}\Bigg]^{-1}$$
$$\times\left[\left(\frac{a_0^2-\beta}{2a_0}\right)^2-\left(\frac{4(a_0+\alpha)^2-\beta}{4(a_0+\alpha)}\right)^2\right]^{-1/2} \tag{48c}$$

where we have used Beta function, which is defined as

$$B(x,y) = \int_0^\infty \frac{z^{y-1}}{(1+z)^{x+y}}\mathrm{d}z \qquad x,y>0.$$

The unnormalized wavefunctions are obtained by using the equations (12), (46) and (47)

$$\chi_1(r,a_0) = -\frac{\hbar}{\sqrt{2m}}\left[(a_0+\alpha)\mathrm{e}^{\alpha r}+\left(\frac{a_0^2-\beta}{2a_0}-\frac{(a_0+\alpha)^2+\beta}{2(a_0+\alpha)}\right)(\mathrm{e}^{\alpha r}-1)+a_0\right]$$
$$\times(\mathrm{e}^{\alpha r}-1)^{((a_0+\alpha)/\alpha)-1}\exp\left[-\frac{(a_0+\alpha)^2+\beta}{2(a_0+\alpha)}r\right] \tag{49a}$$

$$\chi_2(r,a_0) = \frac{\hbar^2}{2m}\left[\frac{2a_0+\alpha}{\mathrm{e}^{\alpha r}+1}+\frac{a_0^2-\beta}{2a_0}-\frac{(a_0+2\alpha)^2+\beta}{2(a_0+2\alpha)}\right]$$
$$\times\left[(a_0+2\alpha)\mathrm{e}^{\alpha r}+\left(\frac{(a_0+\alpha)^2-\beta}{2(a_0+\alpha)}-\frac{(a_0+2\alpha)^2+\beta}{2(a_0+2\alpha)}\right)(\mathrm{e}^{\alpha r}-1)+a_0+\alpha\right]$$
$$\times(\mathrm{e}^{\alpha r}-1)^{(a_0/\alpha)+1}\exp\left[-\frac{(a_0+2\alpha)^2+\beta}{2(a_0+2\alpha)}r\right]$$
$$+\frac{\hbar^2}{2m}\left[(a_0+2\alpha)\mathrm{e}^{\alpha r}+\left(\frac{(a_0+\alpha)^2-\beta}{2(a_0+\alpha)}-\frac{(a_0+2\alpha)^2+\beta}{2(a_0+2\alpha)}\right)(\mathrm{e}^{\alpha r}-1)\right]$$
$$\times\alpha(\mathrm{e}^{\alpha r}-1)^{(a_0/\alpha)+1}\exp\left[-\frac{(a_0+2\alpha)^2+\beta}{2(a_0+2\alpha)}r\right]. \tag{49b}$$

We put $n = n_r + 1$. By using equations (48) and (49) and after algebraic simplification, we obtain the normalized wavefunctions

$$R_{1,0}(r) = \frac{\sqrt{\alpha}}{B((\beta/\alpha^2)-1,3)}\frac{\mathrm{e}^{\alpha r}-1}{r}\mathrm{e}^{-((\alpha^2+\beta)/2\alpha)r} \tag{50a}$$

$$R_{2,0}(r) = \sqrt{\alpha}\left[\frac{3}{2}\alpha\,\mathrm{e}^{\alpha r}+\frac{3}{2}\alpha+\frac{\beta}{4\alpha}-\frac{\beta}{4\alpha}\mathrm{e}^{\alpha r}\right]$$
$$\times\left[B\left(\frac{\beta}{2\alpha^2}-2,5\right)\left[\left(\frac{\alpha}{2}-\frac{\beta}{2\alpha}\right)^2-\left(\alpha-\frac{\beta}{4\alpha}\right)^2\right]^{1/2}\right]^{-1}\frac{\mathrm{e}^{\alpha r}-1}{r}$$
$$\times\mathrm{e}^{-((4\alpha^2+\beta)/4\alpha)r} \tag{50b}$$

$$R_{3,0}(r) = \sqrt{\alpha}\left[rB\left(\frac{\beta}{3\alpha^2}-3,7\right)\left[\left(\frac{4\alpha^2-\beta}{4\alpha}\right)^2-\left(\frac{9\alpha^2-\beta}{6\alpha}\right)^2\right]^{1/2}\right.$$
$$\times\left[\left(\frac{\alpha^2-\beta}{2\alpha}\right)^2-\left(\frac{16\alpha^2-\beta}{8\alpha}\right)^2\right]^{1/2}\Bigg]^{-1}$$
$$\times\left[\left(\frac{3\alpha}{\mathrm{e}^{\alpha r}-1}-\frac{2}{3}\frac{\beta}{\alpha}-\alpha\right)\left(3\alpha\,\mathrm{e}^{\alpha r}-\left(\frac{\alpha}{2}+\frac{5}{12}\frac{\beta}{\alpha}\right)(\mathrm{e}^{\alpha r}-1)+2\alpha\right)\right.$$
$$\left.+3\alpha^2\,\mathrm{e}^{\alpha r}-\left(\frac{\alpha^2}{2}+\frac{5\beta}{12}\right)(\mathrm{e}^{\alpha r}-1)\right](\mathrm{e}^{\alpha r}-1)^2\,\mathrm{e}^{-((9\alpha^2+\beta)/6\alpha)r}. \tag{50c}$$

Special mention may be made of the Hulthen potential. As far as we are aware, the normalized energy eigenfunctions (s state, $l = 0$) have not been explicitly worked out in the literature for this potential.

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